R300 – Advanced Econometric Methods

PROBLEM SET 2 - SOLUTIONS

Due by Mon. October 19

1. Suppose that

$$x_i \sim N(\theta, \sigma_i^2)$$

for known σ_i^2. (Note that, conditional on $\sigma_1^2, \ldots, \sigma_n^2$ the data are not identically distributed. However, we can set this into a random sampling framework if x_i, σ_i^2 are i.i.d. draws from some joint distribution.)

(i) Show that the sample mean

$$\hat{x} = n^{-1} \sum_{i=1}^{n} x_i$$

is unbiased for θ. Derive its sampling variance.

(ii) Show that the estimator

$$\check{x} = \sum_{i=1}^{n} w_i x_i, \qquad w_i = \frac{1/\sigma_i^2}{\sum_{i'=1}^{n} 1/\sigma_{i'}^2}$$

is unbiased. Derive its sampling variance.

(iii) Show that

$$\text{var}_\theta(\check{x}) \leq \text{var}_\theta(\hat{x}).$$

(i) Unbiasedness is immediate by the sample-mean theorem. The variance is

$$\text{var}_\theta(\hat{x}) = \frac{1}{n^2} \sum_{i=1}^{n} \text{var}_\theta(x_i) = \frac{\frac{1}{n} \sum_{i=1}^{n} \sigma_i^2}{n}.$$

(ii) Unbiasedness follows from the fact that the weights sum up to one. The variance is

$$\text{var}_\theta(\check{x}) = \sum_{i=1}^{n} \text{var}_\theta(w_i x_i) = \sum_{i=1}^{n} w_i^2 \, \sigma_i^2 = \frac{1}{\sum_{i=1}^{n} 1/\sigma_i^2}.$$

(iii) We need to show that

$$\frac{\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2}{n} \geq \frac{1}{\sum_{i=1}^{n} 1/\sigma_i^2}.$$

Re-arranging yields the equivalent condition that

$$1 = 1^2 \leq \left(\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2\right)\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sigma_i^2}\right).$$

Also note that

$$1^2 = \left(\frac{1}{n}\sum_{i=1}^{n}\sigma_i\,\frac{1}{\sigma_i}\right)^2,$$

and so the result follows by Cauchy-Shwarz. This exercise is a simple application of Gauss-Markov. Indeed, in the same way it is easy to show that the variance of any linear estimator of the form

$$\sum_i \eta_i\, x_i$$

for weights that satisfy $\sum_i \eta_i = 1$ (otherwise the estimator is not unbiased) is at least as large as the variance of $\check{x}$. Hence, $\check{x}$ is the best linear unbiased estimator of θ.

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2. Consider the finite population of the three numbers 1,2, and 3.

(i) Write down all possible random samples of size two from this population.

(ii) Compute the sample mean in each of these samples.

(iii) Compute the population mean and the mean of the sample mean. What do you conclude?

(iv) Compute the variance of the sample mean. Compare it to the population variance. What do you conclude?

(i) The are 3^2 possible such samples. They are

$$\{1,1\},\ \{1,2\},\ \{1,3\},\ \{2,1\},\ \{2,2\},\ \{2,3\},\ \{3,1\},\ \{3,2\},\ \{3,3\}.$$

(ii) The sample means are

$$1,\ \frac{3}{2},\ 2,\ \frac{3}{2},\ 2,\ \frac{5}{2},\ 2,\ \frac{5}{2},\ 3.$$

(iii) The population mean is

$$\frac{1+2+3}{3} = \frac{6}{3} = 2.$$

The mean of all the sample means is

$$\frac{1+\frac{3}{2}+2+\frac{3}{2}+2+\frac{5}{2}+2+\frac{5}{2}+3}{3^2} = \frac{1+3+2+2+5+2+3}{9} = \frac{18}{9} = 2.$$

The equality is just a manifestation of the unbiasedness of the sample mean, i.e., the expectation of the sample average equals the the population mean.

(iv) The population variance is

$$\frac{(1-2)^2+(2-2)^2+(3-2)^2}{3} = \frac{2}{3}.$$

The squared deviations of the sample means from the population mean are

$$1,\ \frac{1}{4},\ 0,\ \frac{1}{4},\ 0,\ \frac{1}{4},\ 0,\ \frac{1}{4},\ 1.$$

The mean of these values is

$$\frac{3}{9} = \frac{1}{3}.$$

Note that

$$\frac{2/3}{2} = \frac{1}{3}.$$

This is a manifestation of the sample-mean theorem.

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3. The Pareto distribution on the open interval $[m, \infty)$ is the one-parameter distribution with density

$$\frac{\theta m^\theta}{x^{1+\theta}}$$

for $\theta > 0$ (so the minimum value $m \geq 0$ is known here). This distribution is popular for modelling income.

(i) Derive the score and Fisher information for θ.

(ii) Show that

$$y = \log(x/m)$$

follows an exponential distribution with *rate* parameter θ .

(iii) If $y_1, \ldots, y_n$ are random draws from an exponential with rate parameter θ then $u = \sum_{i=1}^n y_i$ follows a Gamma distribution with *shape* and *scale* parameters $n, 1/\theta$. Because n here is an integer the distribution is also called the Erlang distribution. Its density at u is

$$\frac{\theta^n u^{n-1} e^{-\theta u}}{(n-1)!}.$$

Verify this for $n = 2$.

(iv) If u is Gamma distributed as above then

$$v = n/u = n/\sum_i y_i$$

is distributed as Inverse Gamma with *shape* and *scale* parameters $n, n\theta$. Verify this. The Inverse Gamma with shape and scale n, β has density

$$\frac{1}{(n-1)!}\beta^n \frac{1}{v^{n+1}} e^{-\beta/v}$$

(i) The log density is

$$\log\theta + \theta\log m - (1+\theta)\log x$$

and so the score equals

$$\frac{1}{\theta} + \log m - \log x = \frac{1}{\theta} - \log\frac{x}{m}.$$

The Hessian is

$$-\frac{1}{\theta^2} < 0$$

and so the information is simply

$$\frac{1}{\theta^2}.$$

(ii) Let

$$y = \log(x/m).$$

Then the inverse transform is

$$x = me^y$$

and

$$\frac{\partial x}{\partial y} = me^y.$$

The density of y is therefore

$$\frac{\theta m^\theta}{(me^y)^{1+\theta}} me^y = \frac{\theta}{e^{\theta y}} = \theta e^{-\theta y}.$$

This is indeed an exponential with scale θ (and, hence, mean $1/\theta$).

(iii) If y_1 and y_2 are independent exponential their joint density function is

$$\theta^2 e^{-\theta(y_1+y_2)}.$$

Let $u = y_1 + y_2$ and $v = y_2$. The marginal density of u then is

$$\textstyle\int_0^u \theta^2 e^{-\theta u}\, dv = \theta^2 e^{-\theta u} \int_0^u dv = \theta^2 u e^{-\theta u};$$

verify that this function is non-negative on $[0, +\infty]$ and that it integrates to one. This is an Erlang distribution with parameters 2 and θ.

(iv) If u is Erlang then we want the density of

$$v = \frac{n}{u}.$$

The inverse transformation is $u = n/v$ and has derivative

$$-\frac{n}{v^2}.$$

The density of v is thus

$$\frac{1}{(n-1)!}\theta^n \left(\frac{n}{v}\right)^{n-1} e^{-\theta n/v}\, \frac{n}{v^2} = \frac{1}{(n-1)!}\beta^n \frac{1}{v^{n+1}} e^{-\beta/v}$$

for $\beta = n\theta$.

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4. Use the results from the previous question to answer the following questions.

(i) The *maximum likelihood estimator* of θ in the Pareto distribution, based on a random sample of n observations $x_1, \ldots, x_n$, equals

$$\hat{\theta} = \frac{n}{\sum_{i=1}^n \log(x_i/m)}.$$

Is this estimator unbiased?

(ii) Can you derive a first-order approximation (in n) of the bias and variance of $\hat{\theta}$? Does the variance of $\hat{\theta}$ approach the efficiency bound?

(iii) Can you come up with an unbiased estimator? Explain.

(i) Note that

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} \log(x_i/m)} = \left(\frac{\sum_{i=1}^{n} y_i}{n} \right)^{-1}, \qquad y_i = \log(x_i/m).$$

From the previous question we know that the y_i are random draws from an exponential distribution with rate θ. Furthermore, we know that $\hat{\theta}$ has an Inverse Gamma distribution with parameters n and $n\theta$. Hence,

$$E_\theta(\hat{\theta}) = \frac{n\theta}{n-1} > \theta$$

and so the MLE is not unbiased.

(ii) We have

$$E_\theta(\hat{\theta}) = \theta + \frac{\theta}{n-1} = \theta + O(n^{-1}).$$

So the bias vanishes like n^{-1}. For any $n > 2$ we also have

$$\mathrm{var}_\theta(\hat{\theta}) = \frac{n^2\theta^2}{(n-1)^2(n-2)} = \frac{\theta^2}{n} + o(n^{-1}).$$

(iii) An unbiased (bias-corrected) estimator is

$$\frac{n-1}{n}\hat{\theta}.$$

Its variance is

$$\left(\frac{n-1}{n}\right)^2 \mathrm{var}_\theta(\hat{\theta}) = \left(\frac{n-1}{n}\right)^2 \frac{n^2\theta^2}{(n-1)^2(n-2)} = \frac{\theta^2}{n-2} = \frac{\theta^2}{n} + o(n^{-1})$$

and is smaller than that of the MLE yet larger than the parametric efficiency bound.
